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Symmetries of the wave equation in a uniform lattice

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Abstract. The symmetries of discrete versions of a class of equations (that includes Klein–Gordon and wave equations) on a two-dimensional grid are studied. They close the same Lie algebra as the corresponding continuous equations. The relationship to difference special functions is also given.

1. Introduction

Difference equations, as well as differential-difference equations, have recently raised a lot of interest in the physics literature. This is due, in part, to the fact that they constitute a natural way to approach numerically real physical situations, but also because there are many models based on such kinds of equations, for instance applications to dissipative systems [1], nuclear physics [2], and to the study of phonons [3] and magnons [4]. However, some work has already been developed concerning the symmetries of linear difference equations on geometric lattices (q-lattices). In these cases, the relevant algebraic structure is a deformation of the Lie algebra satisfied by the symmetry generators of the corresponding differential equations (indeed, in some cases it is just a q-algebra [5]).

The original motivation of this paper was to consider some fundamental questions raised by the problem of discretizing differential equations:

(i) the analysis of the symmetry algebras emerging when the discretization is uniform, instead of being a q-lattice;

(ii) the use of alternative methods for finding discrete symmetries;

(iii) the application of the discrete symmetries in order to find solutions with specific features, and their relationship to special functions in a discrete variable.

It is necessary to point out that the discretization process for a differential equation involves not only the choice of a discrete lattice, but also the selection of some rules for replacing the differential operators by their discrete analogues. As an example, the operator ∂_z can be changed for a symmetric discrete derivative $(T_z - T_z^{-1})$ or for a directional one $(T_z - 1)$, T_z being the unit translation operator on the lattice (to be described in detail in the next section). The symmetries, as well as their commutations, are remarkably different in these two cases, and accordingly the general properties of the solutions are also modified. We think that it is of interest to get more information about the physical influence that the choice of the operators can have on the symmetries of the solutions (it can be said that, although deformed, the initial algebraic structure of the continuous equation survives in some sense [6]).

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With regard to the nonlinear differential-difference equations, there have recently been some attempts to extend the Lie theory for the symmetries of differential equations to this arena, but the new theory is far from being completed [7–9]. Thus, we think that it is interesting to carry on a study of the symmetries of some concrete difference equations, relevant because of their physical meaning. In addition to their intrinsic interest, the results obtained can be helpful as a guide for more complex equations. In a recent paper [10], the symmetries of the discretized heat equation were considered. Our purpose now is to apply a different procedure in order to look for the symmetries of the one-dimensional wave equation considered in a uniform lattice. This is accomplished in section 2, where it is shown that the resulting symmetries close a Lie algebra. Meixner polynomials appear when the solutions of the wave equation are analysed. Section 3 deals with an interesting limiting case, when one of the discrete variables becomes continuous. In this context the special functions involved are Charlier polynomials. We conclude the paper with some remarks.

2. The wave equation

Let us start by considering the difference equation

$$\left\{\Delta_t^2 - \frac{1}{s^2}\Delta_x^2 + m^2\right\}\Phi(t,x) = 0. \tag{2.1}$$

Here, the variables t and x take values in a uniform lattice: $t = k\tau$, $x = n\sigma$ (τ and σ are the uniform steps; $k, n \in \mathbb{Z}$). In addition, we use the following notation:

$$\Delta_t = \frac{1}{\tau}(T_t - 1) \qquad \Delta_x = \frac{1}{\sigma}(T_x - 1) \tag{2.2}$$

where $m^2 \in \mathbb{R}$. For us, the parameter s^2 takes the values $s^2 = \pm 1$; for other real values of it, we would get essentially the same results (up to a change of scale in the lattice). The translation operators T_t and T_x are defined by

$$T_t\Phi(t,x) = \Phi(t+\tau, x) \qquad T_x\Phi(t,x) = \Phi(t, x+\sigma). \tag{2.3}$$

Of course, if we consider the limit $\tau, \sigma \to 0$, equation (2.1) tends to the one-dimensional Klein–Gordon equation (when $s^2 = +1$):

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{s^2}\frac{\partial^2}{\partial x^2} + m^2\right\}\Phi(t,x) = 0. \tag{2.4}$$

For the sake of clarity, we will pose the problem explicitly in terms of the integer variables k and n, and we will rewrite (2.2) and (2.3) introducing the following notation:

$$\Phi(t,x) = \Phi(k\tau n\sigma) = \phi(k,n) \tag{2.5}$$

$$\Delta_k = \frac{1}{\tau}(T_k - 1) \qquad \Delta_n = \frac{1}{\sigma}(T_n - 1) \tag{2.6}$$

$$T_k\phi(k,n) = \phi(k+1,n) \qquad T_n\phi(k,n) = \phi(k,n+1). \tag{2.7}$$

Then, the initial equation (2.1) can be rewritten as

$$\left\{\Delta_k^2 - \frac{1}{s^2}\Delta_n^2 + m^2\right\}\phi(k,n) = 0. \tag{2.8}$$

Our main objective is the richer structure that appears in the case $m = 0$ (which is precisely the wave equation if $s^2 = 1$), nevertheless, as a first step, we will consider the symmetries appearing when $m \neq 0$. This does not require any extra effort and it will also supply us with a global perspective of this kind of equation.

Here, by a symmetry of an equation we mean any local operator such that when it acts on a solution it gives another solution of the same equation. In our case, these solutions are two-discrete variable functions $\phi(k,n)$ satisfying the linear equation (2.8), that we express in the form

$$E(k,n,T_k,T_n)\phi(k,n) = 0.$$

We shall consider symmetries $S(k,n,T_k,T_n)$ that are made out of the elementary discrete operators k,n,T_k,T_n. Note that, as a consequence, they cannot be thought of as fields to be exponentiated (as is the case for linear differential equations), but just as discrete transformations.

In order to guarantee that an operator S is a symmetry, the commutation

$$ES = \Lambda E \tag{2.9}$$

has to be fulfilled, with Λ a certain operator whose form will depend on S.

For a symmetry S we propose the following ansatz:

$$\begin{aligned} S(k,n,T_k,T_n) = {} & S_0(T_k,T_n) + kS_k(T_k,T_n) + nS_n(T_k,T_n) \\ & + k^2 S_{kk}(T_k,T_n) + knS_{kn}(T_k,T_n) + n^2 S_{nn}(T_k,T_n) + \cdots . \end{aligned} \tag{2.10}$$

In order to apply condition (2.9) on the symmetry S, given by (2.10), one must take into account the basic commutators

$$[T_k,k] = T_k \qquad [T_n,n] = T_n \qquad [T_k,n] = [T_n,k] = 0. \tag{2.11}$$

Then, it is straightforward to arrive at the following results.

(i) If $m \neq 0$, we find three independent non-trivial symmetries:

$$P_k \equiv \Delta_k \qquad P_n \equiv \Delta_n \tag{2.12}$$

$$L \equiv (k\tau)T_k^{-1}\Delta_n + s^2(n\sigma)T_n^{-1}\Delta_k. \tag{2.13}$$

We have proved that when restricting (2.10) up to second order in k and n there are no other independent symmetries. The above operators close the following Lie algebra:

$$[L,P_n] = -s^2 P_k \qquad [L,P_k] = -P_n \qquad [P_k,P_n] = 0. \tag{2.14}$$

Thus, if $s^2 = +1$ we identify this algebra with $iso(1,1)$, while for $s^2 = -1$ it is isomorphic to $iso(2)$.

(ii) If $m = 0$, besides the three symmetries already considered $\{P_k, P_n, L\}$, there are three more, which have the following expressions:

$$D = k\tau T_k^{-1}\Delta_k + n\sigma T_n^{-1}\Delta_n \tag{2.15}$$

$$C_k = k^2\tau^2 T_k^{-2}\Delta_k + s^2 n^2\sigma^2 T_n^{-2}\Delta_k - (k\tau^2 T_k^{-2} + s^2 n\sigma^2 T_n^{-2})\Delta_k + 2kn\tau\sigma T_k^{-1}T_n^{-1}\Delta_n$$

$$C_n = k^2\tau^2 T_k^{-2}\Delta_n - s^2 n^2\sigma^2 T_n^{-2}\Delta_n - (k\tau^2 T_k^{-2} - s^2 n\sigma^2 T_n^{-2})\Delta_n - 2s^2 kn\tau\sigma T_k^{-1}T_n^{-1}\Delta_k.$$

The six generators $\{P_k, P_n, L, D, C_k, C_n\}$ close a Lie algebra whose non-null commutators are given by (2.14) together with

$$\begin{aligned} &[D,P_n] = -P_n \qquad [D,P_k] = -P_k \\ &[P_n,C_n] = 2D \qquad [P_n,C_k] = -2s^2 L \\ &[P_k,C_k] = 2D \qquad [P_k,C_n] = \frac{2}{s^2}L \\ &[D,C_n] = C_n \qquad [L,C_n] = -2s^2 C_k \\ &[D,C_k] = C_k \qquad [L,C_k] = -2C_n. \end{aligned} \tag{2.16}$$

We see that this coincides with the conformal algebra $C(1, 1)$ corresponding to a pseudo-Euclidean space $\mathcal{M}(1, 1)$ if $s^2 = 1$ (isomorphic to $O(2, 2)$), or the conformal $C(2)$ corresponding to the Euclidean space $\mathcal{M}(2)$ (isomorphic to $O(3, 1)$).

Notice that the Lie algebraic structure of the discrete symmetries (2.14) and (2.16) is independent of the lattice steps (τ, σ), and coincides with that associated with the continuous limit equation (2.4). From now on, we will discuss only the case $m = 0$ (which will be referred to, for simplicity, as the wave equation). Once the symmetries are known, we can get solutions of the discrete wave equation among the eigenfunctions of commuting symmetries. Choosing P_k, P_n, we have a family of solutions

$$\phi_\lambda(k, n) = (1 + \lambda\tau)^k (1 + \lambda'\sigma)^n \qquad \lambda' = \pm s\lambda \qquad \lambda \in \mathbb{C} \tag{2.17}$$

corresponding to the eigenvalues

$$P_k\phi_\lambda(k, n) = \lambda\phi_\lambda(k, n) \qquad P_n\phi_\lambda(k, n) = \lambda'\phi_\lambda(k, n). \tag{2.18}$$

Our aim now is to express these solutions in terms of the eigenfunctions of another symmetry operator D. It is easy to show that the polynomial eigenfunctions of D have the form

$$\vartheta_{a,b}(k, n) = \tau^a (k)_a \sigma^b (n)_b \qquad a, b \in \mathbb{Z}^+ \tag{2.19}$$

and satisfy

$$D\vartheta_{a,b}(k, n) = (a + b)\vartheta_{a,b}(k, n) \tag{2.20}$$

where $(p)_c$ is the Pochhammer symbol $(p)_c = p(p-1)\cdots(p-c+1)$. One can also check that the action of the discrete derivatives on these functions $\vartheta_{a,b}(k, n)$ is

$$\Delta_k\vartheta_{a,b}(k, n) = a\vartheta_{a-1,b}(k, n) \qquad \Delta_n\vartheta_{a,b}(k, n) = b\vartheta_{a,b-1}(k, n). \tag{2.21}$$

The expansion we are looking for is given by

$$\phi_\lambda^+(k, n) = (1 + \lambda\tau)^k (1 + \lambda'\sigma)^n = \sum_{a,b\geqslant 0}^{\infty} c_{a,b}\vartheta_{a,b}(k, n) \tag{2.22}$$

where the superscript $+$ on $\phi_\lambda^+(k, n)$ means that we have made the choice of sign in (2.17) $\lambda' = +s\lambda$ (the other possibility gives analogous results). By applying P_k and P_n to equation (2.22), and taking into account (2.18), we find the unknown coefficients

$$\phi_\lambda^+(k, n) = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \sum_{a=0}^{\ell} \binom{\ell}{a} (s)^{\ell-a} \vartheta_{a,\ell-a}(k, n) \equiv \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!} \psi_\ell(k, n; s) \tag{2.23}$$

where we have introduced the polynomials $\psi_\ell(k, n; s)$ of order ℓ in the variables k, n. Let us fix our attention on these polynomials. They have the remarkable property of also being eigenfunctions of D:

$$D\psi_\ell(k, n; s) = \ell\psi_\ell(k, n; s). \tag{2.24}$$

Indeed, if we replace here the explicit form of D given in (2.15), after a few computations, we find a three term recurrence relation:

$$\begin{aligned} &\psi_{\ell+1}(k, n; s) + [\ell(\tau + s\sigma) - (\tau k + s\sigma n)]\psi_\ell(k, n; s) \\ &\qquad + \ell s\tau\sigma[(\ell - 1) - (k + n)]\psi_{\ell-1}(k, n; s) = 0 \end{aligned} \tag{2.25}$$

valid for $\ell \geqslant 1$.

The whole set of polynomials $\{\psi_\ell(k,n;s)\}_{\ell=0}^{\infty}$ constitutes a basis for the support space of a representation of the symmetry algebra. We can easily find the action of the symmetry generators on that basis:

$$P_k\psi_\ell(k,n;s) = \ell\psi_{\ell-1}(k,n;s) \qquad P_n\psi_\ell(k,n;s) = s\ell\psi_{\ell-1}(k,n;s) \tag{2.26}$$

$$C_k\psi_\ell(k,n;s) = \ell\psi_{\ell+1}(k,n;s) \qquad C_n\psi_\ell(k,n;s) = s^2\ell\psi_{\ell+1}(k,n;s) \tag{2.27}$$

$$D\psi_\ell(k,n;s) = \ell\psi_\ell(k,n;s) \qquad L\psi_\ell(k,n;s) = s\ell\psi_\ell(k,n;s). \tag{2.28}$$

Looking at the action of P_k and P_n given in (2.26), we realize that each of the functions $\psi_\ell(k,n;s)$ is a solution of the wave equation. Notice that, since $\{C_k, P_k, D\}$ close an $so(2,1)$ algebra when $s^2 = 1$ ($so(3)$ for $s^2 = -1$), the representation restricted to that set of generators is an $so(2,1)$ representation. However, it is not irreducible: the one-dimensional subspace $\{\psi_0(k,n;s)\}$ is invariant. Irreducibility can only be obtained by going to the quotient space.

There are other useful expressions for this kind of polynomial. For instance, we can show that they can be generated acting on $\psi_0(k,n;s) = 1$:

$$\psi_\ell(k,n;s) = (\tau k T_k^{-1} + s\sigma n T_n^{-1})^\ell 1. \tag{2.29}$$

It is also possible to find a Rodrigues-like formula when $k, n \geqslant 0$ (if $k < 0$ or $n < 0$, it must be conveniently changed):

$$\psi_\ell(k,n;s) = k!n!\tau^{k+\ell}(s\sigma)^{n+\ell}(T_k + T_n)^\ell \frac{1}{(k-\ell)!(n-\ell!)\tau^k(s\sigma)^n}. \tag{2.30}$$

The polynomials we have been considering are closely related to the Meixner polynomials. To show this, recall that the generating function for the Meixner polynomials is [11]

$$(1-z)^{-\gamma-x}\left(1-\frac{z}{c}\right)^x = \sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}M_\ell(x,\gamma,c). \tag{2.31}$$

Comparing with (2.23), the change

$$z = -\lambda\tau \qquad c = \frac{\tau}{s\sigma} \qquad x = n \qquad \gamma = -n-k \tag{2.32}$$

allows us to establish the relationship between both kinds of polynomials (when $k \leqslant 0$). Indeed,

$$\left(\frac{-1}{\tau}\right)^\ell \psi_\ell(k,n;s) = M_\ell\left(n, -k-n, \frac{\tau}{s\sigma}\right). \tag{2.33}$$

Some properties can be translated from one class of polynomials to the other. For example, this is the case for the recurrence (2.25), that with the help of the change (2.32), gives the classical recurrence for the Meixner polynomial:

$$cM_{\ell+1}(x,\gamma,c) - [(c-1)x + \ell(1+c) + \gamma c]M_\ell(x,\gamma,c) - \ell(\ell+\gamma-1)M_{\ell-1}(x,\gamma,c) = 0. \tag{2.34}$$

The same can be done for the Rodrigues formula (2.30) for $k \leqslant 0$.

3. Limiting case

Let us undertake now the limiting process in which the discrete variable $k\tau$ goes to a continuum limit. It is achieved with the conditions

$$k \to \infty \qquad \tau \to 0 \qquad k\tau = t. \tag{3.1}$$

In this way, the initial discrete wave equation (given in (2.1) with $m^2 = 0$) turns into

$$\left\{\partial_t^2 - \frac{1}{s^2}\Delta_n^2\right\}\phi(t,n) = 0. \tag{3.2}$$

The symmetries of the new equation keep the general form presented in (2.14) and (2.15), taking care of making the substitutions: $\Delta_k \to \partial_t, T_k \to 1$. The Lie structure is also preserved. The solutions parametrized by λ behave as follows:

$$\phi_\lambda(k,n) = (1+\lambda\tau)^k(1+\lambda'\sigma)^n \to \mathrm{e}^{\lambda t}(1+\lambda'\sigma)^n \equiv \phi_\lambda(t,n). \tag{3.3}$$

The development of $\phi_\lambda(t,n)$ in terms of D-eigenfunctions is similarly obtained:

$$\phi_\lambda(t,n) = \sum_{\ell=0}^{\infty}\frac{\lambda^\ell}{\ell!}\psi_\ell(t,n;s) \tag{3.4}$$

where the new 'semi-discrete' polynomials $\psi_\ell(t,n;s)$ are defined by

$$\psi_\ell(t,n;s) = \sum_{a=0}^{\ell}\binom{\ell}{a}t^{\ell-a}(s\sigma)^a(n)_a. \tag{3.5}$$

Those polynomials are related to those of Charlier. Recall that Charlier polynomials are defined using the generating function

$$\mathrm{e}^z\left(1-\frac{z}{a}\right)^n = \sum_{\ell=0}^{\infty}\frac{z^\ell}{\ell!}c_\ell(n;a). \tag{3.6}$$

Therefore, we conclude that

$$\psi_\ell(1,n;s) = c_\ell\left(n;\frac{-1}{s\sigma}\right) \equiv c_\ell(n;a). \tag{3.7}$$

In fact, the limit

$$\lim_{k\to\infty,\tau\to0,k\tau=1}\psi_\ell(k,n;s) = \psi_\ell(t=1,n;s) \equiv c_\ell\left(n;\frac{-1}{s\sigma}\right) \tag{3.8}$$

can be shown to be equivalent to the usual one relating Meixner and Charlier polynomials:

$$\lim_{\gamma\to\infty}\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}M_\ell\left(n,\gamma,\frac{\alpha}{\gamma}\right) = c_\ell(n;\alpha) \tag{3.9}$$

putting $\alpha = -1/s\sigma$. The action of generators $\{P_t, P_n, D, C_t, C_n\}$ on $\psi_\ell(t,n;s)$ follows the same rules that we indicated in (2.26)–(2.28). (Note that we have introduced P_t and C_t as the limits of P_k and C_k, respectively. The same applies to $\psi_\ell(t,n;s)$ with respect to $\psi_\ell(k,n;s)$.) In particular, the action of P_n gives

$$\Delta_n\psi_\ell(t,n;s) = \ell s\psi_{\ell-1}(t,n;s). \tag{3.10}$$

This is, in Charlier's terms,

$$(T_n-1)c_\ell(n;a) = -\frac{\ell}{a}c_{\ell-1}(n;a) \tag{3.11}$$

for the value $t = 1$.

To show explicitly the action of the operator D in the limit, observe that it has the form

$$D = (t\partial_t + n\sigma T_n^{-1}\Delta_n). \tag{3.12}$$

Restricting its action to the functions ϕ_λ^+, and using the wave equation, we have at $t = 1$ that

$$D(t = 1) = \left(\frac{1}{s} + \sigma n T_n^{-1}\right)\Delta_n. \tag{3.13}$$

Since

$$D\psi_\ell(1, n; s) = \ell\psi_\ell(1, n; s) \tag{3.14}$$

we obtain the difference equation

$$\left\{\sigma^2 n\nabla_n\Delta_n + \left(\frac{1}{s} - \sigma n\right)\Delta_n\right\}\psi_\ell(1, n; s) = \ell\psi_\ell(1, n; s) \tag{3.15}$$

with $\nabla_n = (1 - T_n^{-1})/\sigma$. This is the same as the Charlier equation:

$$n(T_n - 1)(1 - T_n^{-1})c_\ell(n; a) + (a - n)(T_n - 1)c_\ell(n; a) + \ell c_\ell(n; a) = 0. \tag{3.16}$$

Other expressions for Charlier polynomials derived from (2.29) and (2.30) are

$$\psi_\ell(1, n; s) \equiv c_\ell\left(n; \frac{-1}{s\sigma}\right) = (1 + s\sigma n T_n^{-1})^\ell 1 \tag{3.17}$$

and the Rodrigues formula

$$\psi_\ell(1, n; s) = n!(s\sigma)^{\ell+n}(1 + T_n)^\ell \; \frac{1}{(n - \ell)!(s\sigma)^n}. \tag{3.18}$$

4. Concluding remarks

We have computed the symmetry algebra corresponding to the discrete wave equation, obtaining as the main result that it coincides with that of the continuous case: the conformal Lie algebra. (This is true at least up to the second order; for higher orders it seems that we would obtain the same infinite-dimensional Lie algebra as that in the corresponding differential equation.) We have restricted ourselves to one spatial plus one time dimension, but it is clear that increasing space dimensions would supply us with the corresponding conformal algebra.

As in the continuous case, the symmetries of discrete equations can be used in order to get a class of solutions that are invariant under subalgebras of the symmetry operators. For instance, the functions ϕ_λ given in (2.17) were obtained as eigenfunctions of the Abelian subalgebra $\langle P_k, P_n\rangle$, while the functions ψ_ℓ defined by (2.23) are solutions invariant under the subalgebra $\langle D, L\rangle$. However, in general, the use of symmetries does not allow for the separation of variables in the original equation. This is in striking contrast to the continuous case, and is a strong limitation due to the underlying grid.

We have shown that the solutions ϕ_λ can be expanded using the polynomial eigenfunctions ψ_ℓ of the dilation generator D, and, hence, the former play the role of generating functions of the latter. The polynomials ψ_ℓ are closely related to classical special functions: Meixner and Charlier polynomials. This relationship was to be expected: Meixner and Charlier polynomials obey difference equations similar to our original equation. Nevertheless, we must point out that the aforementioned solutions depend on both variables, k and n, and cannot be separated by symmetry reduction as in the continuous case. Thus, the connection with orthogonal polynomials has been accomplished by imposing a certain condition on these two variables.

We believe that a further investigation in this line could throw some light on the link between difference symmetries and difference special functions. Our guess is that other special functions of this type (Krawtchouk, Hahn, and so on) will appear in the study of the symmetries of other difference equations set up for physical problems [12].

Another important point to be mentioned is that one would like to design a general procedure to discretize a large class of differential equations in such a way that their symmetries should be conserved in the process. The mere substitution of the derivatives by discrete derivatives can be seen only as a first approximation, but obviously it does not seem to be a fully satisfactory answer to this program. The same point remains open for the q-discretizations (the only difference here is the replacement of the Lie algebra by a q-algebra). Some work in that direction is now in progress.

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